$$x_3 = \frac{\sqrt{a}+\sqrt{b-\sqrt{c}}}{\sqrt{a}-\sqrt{b-\sqrt{c}}} = \frac{1}{x_4},$$

with $a = 2b$ and with $c = b^2 - 4$.

In the given equation $4b^2 = 17$. Then

$$b^2 = \frac{17}{4}, \quad b = \frac{\sqrt{17}}{2} > 2, \ a = 2b = \sqrt{17}, \ c = b^2 - 4 = \frac{1}{4}.$$

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Bruno Salgueiro Fanego, Viveiro, Spain; G.C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Boris Rays, Brooklyn, NY; Henry Ricardo, New York Math Circle, NY. Toshihiro Shimizu, Kawasaki, Japan; Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Albert Stadler, Herrliberg, Switzerland; (David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

- **5380:** *Proposed by Arkady Alt, San Jose, CA*

Let $\Delta(x,y,z) = 2(xy+yz+xz) - (x^2+y^2+z^2)$ and a,b,c be the side-lengths of a triangle ABC. Prove that

$$F^2 \geq \frac{3}{16} \cdot \frac{\Delta(a^3,b^3,c^3)}{\Delta(a,b,c)},$$

where F is the area of $\triangle ABC$.

Solution 1 by Toshihiro Shimizu, Kawasaki, Japan

From the Heron's formula,

$$\begin{aligned} F^2 &= \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{16} \\ &= \frac{\Delta(a^2,b^2,c^2)}{16} \end{aligned}$$

Thus, it suffices to show that $\Delta(a^2,b^2,c^2)\Delta(a,b,c) - 3\Delta(a^3,b^3,c^3) \geq 0 \quad (\heartsuit)$. The (l.h.s) can be written as

$$\sum_{cyc}(a-b)(a-c)q(a,b,c),$$

where $q(a,b,c) = 4a^4 + 2a^3(b+c) + a^2(b-c)^2 \geq 0$. Moreover, since

$$q(a,b,c) - q(b,c,a) = (a-b)(bc^2 + ac^2 + 2b^2c + 2a^2c + 4b^3 + 6ab^2 + 6a^2b + 4a^3),$$

the relation, which is larger $q(a,b,c)$ or $q(b,c,a)$, depends on the value of a or b. Without loss of generality, we assume $a \geq b \geq c$. Then, $q(a,b,c) \geq q(b,c,a) \geq q(c,a,b)$. Thus,

$$\sum_{cyc}(a-b)(a-c)q(a,b,c) = (a-b)\big((a-c)q(a,b,c)-(b-c)q(b,c,a)\big)+q(c,a,b)(a-c)(b-c)$$
$$\geq 0.$$

Therefore, (♡) is true.
Note: It is similar to the proof of Schur's inequality. It seems that (♡) is valid for any a,b,c, even if the constraint that a,b,c are the side-lengths of a triangle is not satisfied.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

Denote by s the semiperimeter of the triangle put $s_a = s-a, s_b = s-b, s_c = s-c$. By the triangle inequality, $s_a \geq 0, s_b \geq 0, s_c \geq 0$. Also $a = s_b + s_c,\ b = s_c + s_a,\ c = s_a + s_b$. Furthermore, we note that

$$\Delta(a,b,c) = \Delta\left(s_b+s_c, s_c+s_a, s_a+s_b\right) = 4\left(s_a s_b + s_b s_c + s_c s_a\right) \geq 0.$$

By Heron's formula $F^2 = s\cdot s_a\cdot s_b\cdot s_c = \left(s_a+s_b+s_c\right) s_a\cdot s_b\cdot s_c$.
Therefore we need to prove that

$$64\left(s_a+s_b+s_c\right)\cdot s_a\cdot s_b\cdot s_c\left(s_a s_b+s_b s_c+s_c s_a\right) \geq 3\Delta\left(\left(s_b+s_c\right)^3,\ \left(s_c+s_a\right)^3,\ \left(s_a+s_b\right)^3\right)$$

which is equivalent to

$$27\sum_{symm} s_a^4 s_b^2 + 21\sum_{symm} s_a^3 s_b^3 + 5\sum_{symm} s_a^2 s_b^2 s_c^2 \geq 27\sum_{symm} s_a^4 s_b s_c + 26\sum_{symm} s_a^3 s_b^2 s_c \qquad (1)$$

(as is seen by simply multiplying out).
By Schur's inequality

$$\sum_{cycl} s_a s_b\left(s_a s_b - s_b s_c\right)\left(s_a s_b - s_c s_a\right) \geq 0$$

which is equivalent to

$$\sum_{symm} s_a^3 s_b^3 + \sum_{symm} s_a^2 s_2^2 s_c^2 \geq 2\sum_{symm} s_a^3 s_b^2 s_c \qquad (2)$$

(as is seen again by multiplying out).
We have the following inequalities

$$5\sum_{symm} s_a^3 s_b^3 + 5\sum_{symm} s_a^2 s_b^2 s_c^2 \quad\geq\quad 10\sum_{symm} s_a^3 s_b^2 s_c, \quad (\text{by}(2)),$$

$$27\sum_{symm} s_a^4 s_b^2 \quad\geq\quad 27\sum_{symm} s_a^4 s_b s_c,\ \text{by Muirhead's inequality},$$

$$16\sum_{symm} s_a^3 s_b^3 \quad\geq\quad 16\sum_{symm} s_a^3 s_b^2 s_c,\ \text{by Muirhead's inequality}.$$

(1) follows by adding these three inequalities.

Solution 3 by proposer

Let $s := \dfrac{t_1+t_2+t_3}{2}$. Since $t_i < s, i = 1,2,3$ (triangle inequalities) then our problem is:
Find $\max s$ for which there are positive integer numbers
t_1, t_2, t_3 satisfying $t_i \le \min\{a_i, s-1\}, i = 1,2,3,\ t_1+t_2+t_3 = 2s$.
First note that $s \ge 3, t_i \ge 2, i = 1,2,3$. Indeed, since $t_i \le s-1$, then
$1 \le s - t_i, i = 1,2,3$ and,
therefore, $t_1 = 2s - t_2 - t_3 = (s-t_2) + (s-t_3) \ge 2$. Cyclic we obtain $t_2, t_3 \ge 2$. Hence,
$2s \ge 6 \iff s \ge 3$.

Since $t_3 = 2s - t_1 - t_2,\ 2 \le t_3 \le \min\{a_3, s-1\}$
then $1 \le 2s - t_1 - t_2 \le \min\{a_3, s-1\} \iff$
$\max\{2s - t_1 - a_3, s+1-t_1\} \le t_2 \le 2s-1-t_1$ and, therefore, for t_2 we obtain the inequality

(1) $\qquad \max\{2s-t_1-a_3, s+1-t_1, 2\} \le t_2 \le \min\{2s-1-t_1, a_2, s-1\}$
with conditions of solvability :

$$\textbf{(2)}\quad \begin{cases} 2s-t_1-a_3 \le s-1 \\ 2s-t_1-a_3 \le a_2 \\ s+1-t_1 \le a_2 \\ 2 \le 2s-1-t_1 \end{cases} \iff \begin{cases} s+1-a_3 \le t_1 \\ 2s-a_2-a_3 \le t_1 \\ s+1-a_2 \le t_1 \\ t_1 \le 2s-3 \end{cases}.$$

Since $s-1 \le 2s-3$ then **(2)** together with $2 \le t_1 \le \min\{a_1, s-1\}$ it gives us bounds for t_1 :

$$\textbf{(3)}\quad \max\{s+1-a_3, 2s-a_2-a_3, s+1-a_2, 2\} \le t_1 \le \min\{a_1, s-1\}.$$

Since $2 \le a_i, i = 2,3$ then $s+1-a_2 \le s-1, s+1-a_3 \le s-1$ and solvability condition for **(3)** becomes

$$s+1-a_3 \le a_1 \iff s \le a_1+a_3-1, 2s-a_2-a_3 \le a_1 \iff s \le \left\lfloor \frac{a_1+a_2+a_3}{2} \right\rfloor,$$
$$s+1-a_2 \le a_1 \iff s \le a_1+a_2-1, 2s-a_2-a_3 \le s-1 \iff s \le a_2+a_3-1.$$

Thus, $s^* = \min\left\{ \left\lfloor \dfrac{a_1+a_2+a_3}{2} \right\rfloor, a_1+a_2-1, a_2+a_3-1, a_3+a_1-1 \right\}$ is the largest value of integer semiperimeter.

Solution 4 by Andrea Fanchini, Cantú, Italy

We know that
$$F^2 = s(s-a)(s-b)(s-c)$$
where s is the semiperimeter of $\triangle ABC$.
Now making the substitutions and clearing the denominators we have to prove

$$16s(s-a)(s-b)(s-c)\left[2(ab+bc+ca) - (a^2+b^2+c^2)\right] \ge 3\left[2(a^3b^3+b^3c^3+c^3a^3) - (a^6+b^6+c^6)\right]$$

now we make the following substitutions (with $x, y, z > 0$)

$$a = y+z, \quad b = z+x, \quad c = x+y$$

and expanding out into symmetric sums the given inequality yields
LHS:

$$27(x^4y^2 + x^4z^2 + y^4z^2 + x^2y^4 + x^2z^4 + y^2z^4) + 42(x^3y^3 + y^3z^3 + x^3z^3) +$$
$$+6(x^3yz^2 + x^3y^2z + x^2y^3z + xy^3z^2 + x^2yz^3 + xy^2z^3) + 78x^2y^2z^2$$

RHS:

$$38(x^4yz + xy^4z + xyz^4)$$

so it remains to prove that

$$27(x^4y^2 + x^4z^2 + y^4z^2 + x^2y^4 + x^2z^4 + y^2z^4) \geq 38(x^4yz + xy^4z + xyz^4)$$

or

$$27[4, 2, 0] \geq 19[4, 1, 1]$$

which is true because

$$19[4, 2, 0] \succ 19[4, 1, 1]$$

it follows from Muirhead's Theorem, q.e.d.

Solution 5 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

As well known $F^2 = s(s-a)(s-b)(s-c)$ and $s = (a+b+c)/2$. Upon setting
$a = y+z$, $b = x+z$, $c = x+y$,
the inequality becomes

$$\sum_{\text{sym}} \left(27x^4y^2 + 21(xy)^3 + 5(xyz)^2\right) \geq \sum_{\text{sym}} \left(27x^4yz + 26x^3y^2z\right).$$

The third degree Schür inequality is

$$(a^3 + b^3 + c^3) + 3abc \geq \sum_{\text{sym}} a^2b,$$

which applied to the triple $(xy), (yz), (zx)$, yields

$$5\sum_{\text{sym}}(xy)^3 + 5\sum_{\text{sym}}(xyz)^2 \geq 10\sum_{\text{sym}} x^3y^2z.$$

The inequality becomes

$$\sum_{\text{sym}} \left(27x^4y^2 + 16(xy)^3\right) \geq \sum_{\text{sym}} \left(27x^4yz + 16x^3y^2z\right),$$

and the proof is complete upon observing that by the AGM we have

$$x^4y^2 + x^4z^2 \geq 2x^4yz, \qquad (xy)^3 + (xy)^3 + (xz)^3 \geq 3x^3y^2z.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Moti Levi, Rehovot, Israel, and Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania